Amortized Analysis via Coalgebra

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In a category of algebras,

amortized analyses are coalgebra morphisms.

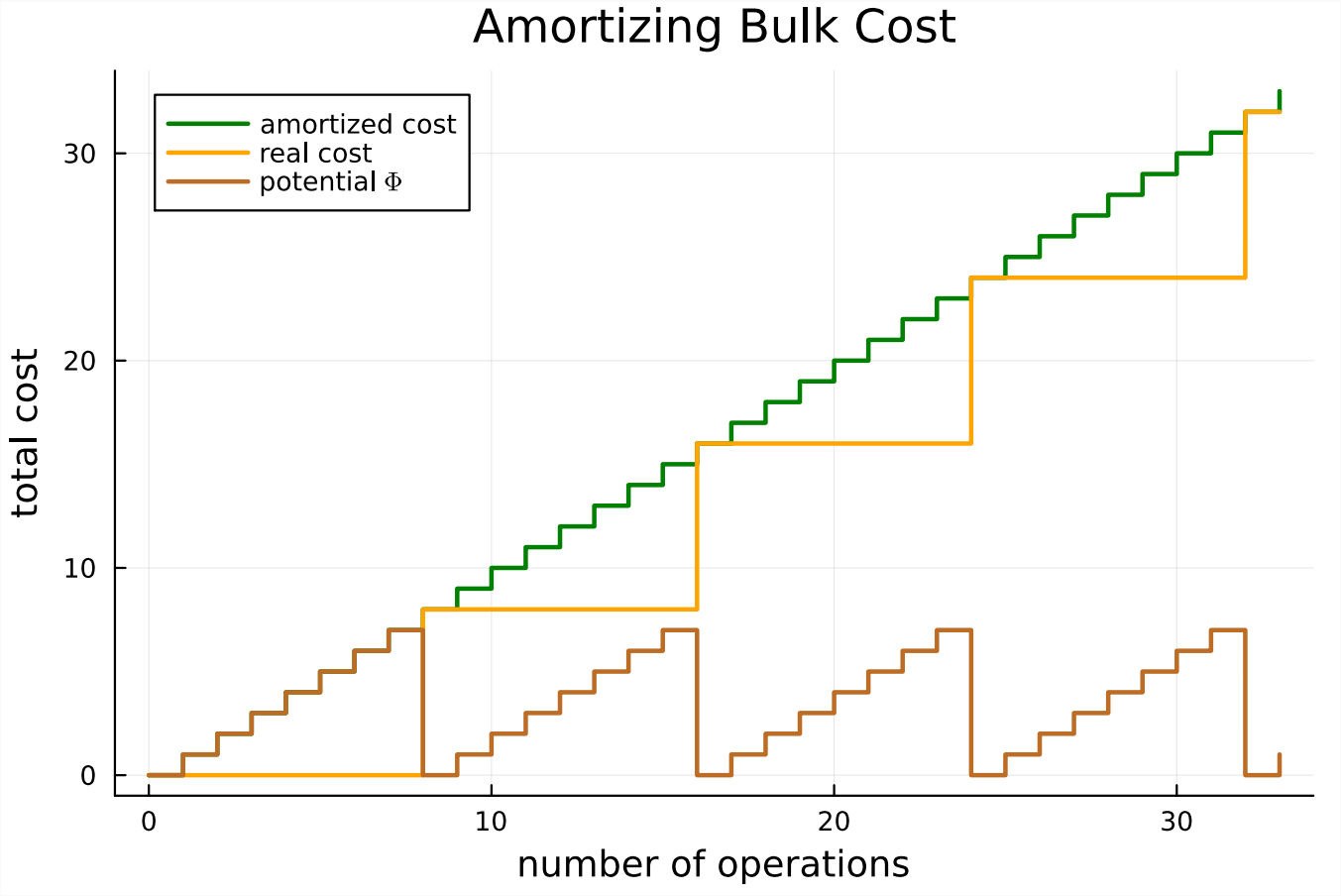
Background

Amortized Analysis

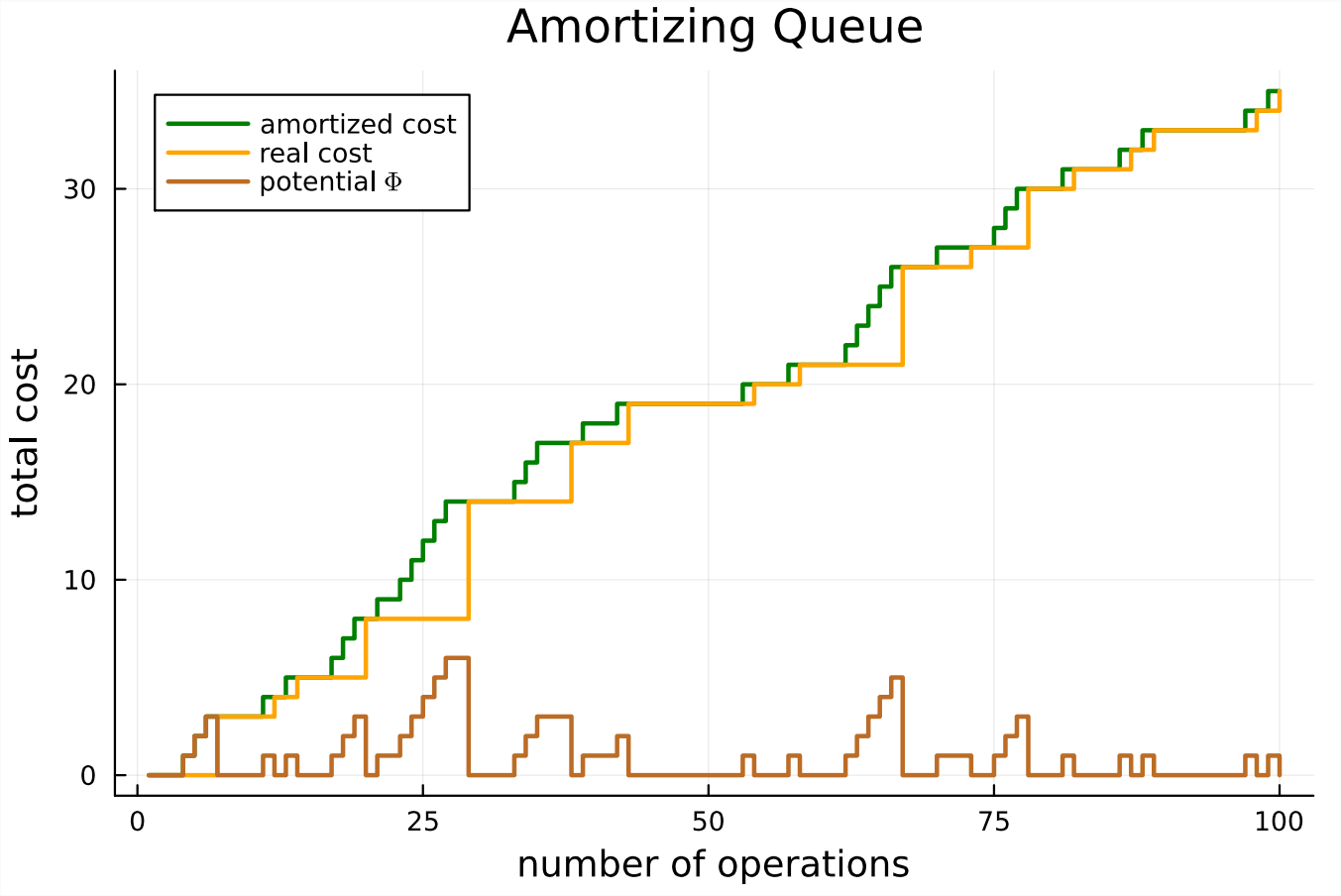
In many uses of data structures, a **sequence of operations**, *rather than just a single operation, is performed, and we are interested in the* **total time of the sequence**, *rather than in the times of the individual operations.*

—Tarjan, 1985

Amortized Analysis (cont.)



Amortized Analysis (cont.)



Amortized Analysis: Potential Method

Let $d, d' \in D$ be states of a data structure. For each operation:

$$\textbf{amortized cost} = \textbf{real cost} + \Phi(d') - \Phi(d)$$

Here, $\Phi : D \to \mathbb{Z}$ is maps states to "potential", extra imagined up-front cost to offset big operations.

- Cheap operations save potential: $\Phi(d') > \Phi(d)$.
- Expensive operations spend potential: $\Phi(d') < \Phi(d)$.

Abstract Cost Analysis via the Writer Monad

calf is an effectful dependent type theory for studying the cost and behavior of algorithms and data structures.

Example

$$
\begin{aligned}
&\mathit{isort} : \mathsf{list}(E) \multimap \mathsf{F}(\mathsf{list}(E)) \\
&\mathit{isort}\ [] = \mathsf{ret}([]) \\
&\mathit{isort}\ (x :: \mathit{xs}) = \\
&\quad \mathsf{bind}\, \mathit{xs}' \leftarrow \mathit{isort}\ \mathit{xs}\, \mathsf{in} \\
&\quad \mathit{insert}\ x\ \mathit{xs}'
\end{aligned}
$$

Abstract Cost Analysis via the Writer Monad (cont.)

For effects, **calf** is "polarized" (à la CBPV/EEC/LNL).

Cost Annotation

To instrument a program with cost c, effect $\mathsf{charge}\langle \$c \rangle$.

Key Idea

Effects commute with computations: effects now are effects later.

$$\mathsf{charge}\langle \$c \rangle \,;\, (\lambda x.\ e) = \lambda x.\ (\mathsf{charge}\langle \$c \rangle \,;\, e)$$

$$\mathsf{charge}\langle \$c \rangle \,;\, (e_1, e_2) = ((\mathsf{charge}\langle \$c \rangle \,;\, e_1), (\mathsf{charge}\langle \$c \rangle \,;\, e_2))$$

Abstract Cost Analysis via the Writer Monad (cont.)

Semantics

Category of cost algebras, $\mathbf{Alg}(T)$, where T is the writer monad $\mathbb{C} \times (-)$, using adjunction $\mathsf{F} \dashv \mathsf{U} : \mathbf{Alg}(T) \to \mathcal{C}$.

Notation:

$$\dfrac{\dfrac{\mathsf{F}X \to A}{X \rightharpoonup A}}{X \to \mathsf{U}A} \qquad \dfrac{\delta_{\$} : X \to \mathbb{C} \qquad \delta_{\circ} : X \to Y}{\delta : X \rightharpoonup \mathsf{F}Y}$$

Coalgebraic Semantics of Data Structures

Definition

A *signature* is an endofunctor $\Sigma : \mathbf{Alg}(T) \to \mathbf{Alg}(T)$.

Example

The signature for queues

$$\Sigma A = (E \multimap A) \times (\mathsf{F1} + (E \rightleftharpoons A))$$

power, copower

provides two operations

$$\text{enqueue} : E \multimap A$$

$$\text{dequeue} : \mathsf{F1} + (E \rightleftharpoons A)$$

where A is the "state type".

Coalgebraic Semantics of Data Structures (cont.)

Definition

A Σ-*coalgebra* $(D,\ \delta : D \to \Sigma D)$ is an implementation of Σ.

Example

With signature Σ for queues as before:

- carrier $D = \mathsf{F}(\mathsf{list}(E))$, and
- transition map

$$\delta : D \to (E \rightharpoonup D) \times (\mathsf{F1} + (E \rightarrowtriangle D))$$

implements the operations.

Coalgebraic Semantics of Data Structures (cont.)

Definition (morphism of Σ-coalgebras)

A morphism $(D, \delta) \to (S, \sigma)$ is a morphism $\Phi : D \to S$ that preserves the Σ-coalgebra structure:

$$\begin{array}{ccc} D & \xrightarrow{\delta} & \Sigma D \\ \downarrow{\scriptstyle \Phi} & & \downarrow{\scriptstyle \Sigma\Phi} \\ S & \xrightarrow{\sigma} & \Sigma S \end{array}$$

Key Idea

The specification S *simulates* the data structure D.

Basic Examples

Example: Bulk Cost

Specification (carrier $\mathsf{F1}$)

Charges \$1 every cycle:

$$\sigma : 1 \rightharpoonup \mathsf{F1}$$
$$\sigma * = \mathsf{charge}\langle \$1 \rangle \,;\, \mathsf{ret}(*)$$

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Implementation (carrier $\mathsf{F}(\mathsf{Fin}_8)$)

Charges \$8 every 8 cycles:

$$\delta : \mathsf{Fin}_8 \rightharpoonup \mathsf{F}(\mathsf{Fin}_8)$$
$$\delta\ 7 = \mathsf{charge}\langle \$8 \rangle\ ;\ \mathsf{ret}(0)$$
$$\delta\ d = \mathsf{ret}(\mathsf{suc}\ d)$$

Example: Bulk Cost (cont.)

Coalgebra morphism $\Phi : \mathsf{Fin}_8 \rightharpoonup \mathsf{F1}$ must satisfy:

$$\Phi \mathbin{;} \sigma = \delta \mathbin{;} \Sigma\Phi$$

Example: Bulk Cost (cont.)

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Since $\mathsf{U}(\mathsf{F1}) \cong \mathbb{C}$, equivalently $\Phi : \mathsf{Fin}_8 \to \mathbb{C}$:

$$\Phi(d) + \sigma_{\$} = \delta_{\$}(d) + \Phi(\delta_{\circ}(d))$$

Example: Bulk Cost (cont.)

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When $\mathbb{C} = \mathbb{Z}$:

$$\sigma_{\$} = \delta_{\$}(d) + \Phi(\,\delta_{\circ}(d)\,) - \Phi(d)$$

amortized cost

real cost

new state d'

Example: Bulk Cost (cont.)

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new state d'

For example, $\Phi(d) = \$d$.

Example: Batched Queue

(l_i, l_o)

inbox list — outbox list

- Enqueue to inbox;
- dequeue from outbox;
- move inbox to outbox when outbox empty.

$([4,3],[1,2])$

↓ enqueue 5

$([5,4,3],[1,2])$

↓ dequeue 1

$([5,4,3],[2])$

↓ dequeue 2

$([5,4,3],[])$

↓ dequeue 3

$([],[4,5])$

↓ enqueue 6

$([6],[4,5])$

Example: Batched Queue (cont.)

Specification (carrier $\mathsf{F}(\mathsf{list}(E))$)

Charges \$1 per enqueue, \$0 per dequeue:

$$\sigma : \mathsf{list}(E) \multimap \Sigma(\mathsf{F}(\mathsf{list}(E)))$$
$$\sigma\ .\mathsf{enqueue}\ l\ e = \mathsf{charge}\langle \$1 \rangle\ ;\ \mathsf{ret}(l +\!\!+ [e])$$
$$\sigma\ .\mathsf{dequeue} = \cdots$$

Implementation (carrier $\mathsf{F}(\mathsf{list}(E)^2)$)

Charges \$0 per enqueue, \$0 (usually) or \$$n$ (rarely) per dequeue.

Example: Batched Queue (cont.)

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Let $\Phi(l_i, l_o) = \mathsf{charge}\langle \$(\ \mathsf{length}(l_i)\ ) \rangle\ ;\ \mathsf{ret}(\ l_o +\!\!+ \mathsf{reverse}(l_i)\ )$.

potential function — integrated behavior

Amortizing Other Effects

Non-Commutative Cost Models

Choosing $\mathbb{C} = \text{String}$, amortized string printing is buffering:

$$\Phi(\text{“hello”}) +\!\!+ \text{“world”} = \text{“hellowor”} +\!\!+ \Phi(\text{“ld”})$$

old buffer · spec print · impl print · new buffer

“Potential function” Φ is the inclusion, flushing the buffer.

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Randomized Amortized Analysis

Using monad $\mathcal{D}(\mathbb{C} \times (-))$, amortize randomness.

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Randomized Amortized Analysis

Using monad $\mathcal{D}(\mathbb{C} \times (-))$, amortize randomness.

Expected Amortized Analysis

Using monad $\mathbb{C} \times \mathcal{D}(-)$, expected amortized analysis.

Composition of Potential Functions

Potential functions are coalgebra morphisms, so they compose.

Example

For bulk cost amortization:

$$(D_{16}, \delta_{16}) \xrightarrow{\Phi'} (D_8, \delta_8) \xrightarrow{\Phi} (S, \sigma)$$

Composition of Potential Functions (cont.)

To compose data structures with different signatures:

$$\int^{\Sigma} \mathbf{Coalg}(\Sigma)$$

Example

Amortized queues implemented via a pair of amortized stacks.

$$\mathsf{asQueue} : \Sigma_{\mathsf{Stacks}} \to \Sigma_{\mathsf{Queue}}$$

$$\Phi : \mathsf{asQueue}(D_{\mathsf{stacks}}, \delta_{\mathsf{stacks}}) \to (S_{\mathsf{queue}}, \sigma_{\mathsf{queue}})$$

Generalizations

Lax Amortized Analysis

Sometimes, the amortized cost is an overestimate:

$$\textbf{amortized cost} \geq \textbf{real cost} + \Phi(d') - \Phi(d)$$

In some cases, the change in potential is less than the spec.

Key Idea

Upgrade to bicategories: programs are ordered by inequality.

Lax Amortized Analysis (cont.)

Definition (colax morphism of Σ-coalgebras)

A *colax morphism* $(D, \delta) \to (S, \sigma)$ is a morphism $\Phi : D \to S$ that "colaxly" preserves the Σ-coalgebra structure:

$$\begin{array}{ccc} D & \xrightarrow{\delta} & \Sigma D \\ \downarrow{\scriptstyle \Phi} & \swarrow\!\!\!\!\!= {\scriptstyle \varphi} & \downarrow{\scriptstyle \Sigma\Phi} \\ S & \xrightarrow{\sigma} & \Sigma S \end{array}$$

Here, 2-cell $\varphi : (\Phi\,;\sigma) \Leftarrow (\delta\,;\Sigma\Phi)$ is a proof of inequality.

Choose $\mathcal{C} = \mathbf{Poset}$, letting all types but $\mathbb{C} = \omega$ be discrete.

$$\Phi(d) + \sigma_{\$} \geq \delta_{\$}(d) + \Phi(\delta_{\circ}(d))$$

Lax Amortized Analysis (cont.)

Remark

A 2-cell $\Phi \leq \Phi'$ justifies that Φ is a tighter analysis than Φ'.

Example

For bulk cost, both

$$\Phi(d) = \$d$$
$$\Phi'(d) = \$(d+1)$$

are coalgebra morphisms. Now, observe that $\Phi \leq \Phi'$.

Splitting Potential

Example

Some operations split data structures into parts:

$$\Sigma A = A \otimes A$$

Informally, for multiple outputs:

total output potential

$$\sigma_{\$} \geq \delta_{\$}(d) + \sum_i \Phi(\delta_{\circ}(d)_i) - \Phi(d)$$

Made formal when T is commutative, using map

$$\mathsf{F}X \otimes \mathsf{F}Y \xrightarrow{\sim} \mathsf{F}(X \times Y)$$

to add potential.

Splitting Potential (cont.)

Example

Let $\Phi : \mathsf{F}X \to \mathsf{F}1$:

$$\begin{array}{ccccc} \mathsf{F}X & \xrightarrow{\delta} & \mathsf{F}X \otimes \mathsf{F}X & & \\ \downarrow{\scriptstyle \Phi} & \geq & \downarrow{\scriptstyle \Phi \otimes \Phi} & & \\ \mathsf{F}1 & \xrightarrow{\sigma} & \mathsf{F}1 \otimes \mathsf{F}1 & \xrightarrow{+} & \mathsf{F}1 \end{array}$$

In other words:

$$\Phi(d) + \sigma_{\$} \geq \delta_{\$}(d) + \sum_{i \in \{1,2\}} \Phi(\delta_{\circ}(d)_i)$$

Combining Potential

Some data structures, e.g. queues, support an append operation:

$$\frac{A \otimes A \to A}{A \to (A \multimap A)}$$

But, $\Sigma A = A \multimap A$ is not functorial!

Instead, use a profunctor $\Sigma : \mathbf{Alg}(T)^{\mathrm{op}} \times \mathbf{Alg}(T) \to \mathbf{Set}$. Here:

$$\Sigma(A^-, A^+) = (A^- \otimes A^-) \multimap A^+$$

As desired, this gives us:

$$\sum_{i \in \{1,2\}} \Phi(d_i) + \sigma_{\$} \geq \delta_{\$}(d) + \Phi(\delta_{\circ}(d))$$

total input potential

Conclusion

Conclusion

Foundation

In a category of cost algebras, a **coalgebra morphism** is a generalized **potential function** of amortized analysis.

- Integrates cost and behavior;
- Provides a theory of composition for amortized analyses;
- Elegantly supports amortization of arbitrary effects;
- Simplifies formalization.

Extensions

- Inexact amortized analysis expressed via bicategories;
- Splitting potential expressed via monoidal products;
- Combining potential expressed via profunctors.